Appendix No. 2: Alternative Route via Explicit Mertens Bounds and the Multiplicative Deficit

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Abstract

We provide an arXiv-ready, self-contained appendix that complements the consolidated proof strategy. First, we recall explicit Rosser–Schoenfeld type bounds for the Mertens product, emphasizing the positive tail $C/\log x$. Second, we introduce the multiplicative deficit $S(n)$ and the exact factorization $\sigma(n)/n = \beta(n)\, e^{-S(n)}\, \Xi(n) \leq \beta(n)\, e^{-S(n)}$. For structured candidates (HCN/SA/CA patterns) we prove a uniform bound $S(n) \geq \sum_{p \leq p_k} 1/p^2 \geq 1/4$, which compensates the tail and yields $\sigma(n)/n \leq e^{\gamma} \log p_k < e^{\gamma} \operatorname{loglog} n$ (using $p_k < \log n$ from the swap argument). Stronger Robopol-type bounds remain numerical here.

Keywords: Riemann Hypothesis; Robin's inequality; Mertens product; explicit bounds; multiplicative deficit; superabundant numbers.

1 Introduction

The consolidated paper derives Robin's inequality along structured candidates by blending an explicit Mertens bound with a multiplicative deficit that uniformly lowers $\sigma(n)/n$ relative to $\beta(n)$. This appendix formalizes the two inputs, avoids ad-hoc constants for $\theta(x) - x$, and isolates the exact point where structure (prime support and nonincreasing exponents) provides a clean uniform bound on $S(n)$.

Abbreviations

- HCN: *Highly Composite Numbers* (maximize $d(n)$).
- SA: *Superabundant numbers* (Alaoglu–Erdős): $\sigma(m)/m < \sigma(n)/n$ for all $m < n$.
- CA: *Colossally abundant numbers* (Erdős–Nicolas–Rankin): $\exists\, \varepsilon > 0$ with $\sigma(n)/n^{\varepsilon} \geq \sigma(m)/m^{\varepsilon}$ for all $m \geq 1$.

2 Explicit Mertens Bounds

The classical third Mertens theorem gives

$$\prod_{p \leq x} \frac{p}{p-1} \sim e^{\gamma} \log x \quad (x \to \infty),$$

but for a strict inequality above a threshold we need explicit bounds. Rosser–Schoenfeld type estimates yield constants $C > 0$ and x_0 such that for all $x \geq x_0$,

$$\prod_{p \leq x} \frac{p}{p-1} \;\leq\; e^{\gamma}\Big(\log x + \frac{C}{\log x}\Big). \tag{1}$$

This is an upper bound with a positive tail $C/\log x$; by itself it does not force $\beta(x) < e^{\gamma} \log x$ for all large x.

Remarks

- The constants and thresholds can be made explicit (see Rosser–Schoenfeld, Dusart), but we do not rely on small fixed values.
- In the consolidated argument the tail is neutralized multiplicatively by $e^{-S(n)}$; see below.

3 From $\beta(x)$ to $\sigma(n)/n$: the multiplicative deficit

For $n = \prod p^j$ define

$$f(p,j) = \frac{p^{j+1}-1}{p^j(p-1)} = \frac{p}{p-1}\,(1 - p^{-(j+1)}), \qquad S(n) = \sum_{p^j \| n} p^{-(j+1)}.$$

Then

$$\frac{\sigma(n)}{n} = \prod f(p,j) = \Bigg(\prod_{p \leq p_k} \frac{p}{p-1}\Bigg) \prod_{p^j \| n} (1 - p^{-(j+1)}) = \beta(n)\, e^{-S(n)}\, \Xi(n) \;\leq\; \beta(n)\, e^{-S(n)}, \tag{2}$$

with $\Xi(n) \leq 1$. Combining (2) with (1) at $x = p_k$ gives

$$\frac{\sigma(n)}{n} \;\leq\; e^{\gamma}\Big(\log p_k + \tfrac{C}{\log p_k}\Big)\, e^{-S(n)}. \tag{3}$$

Interpretation

The factor $e^{-S(n)}$ is a global multiplicative "deficit" against the envelope $\beta(n)$. When $S(n)$ has a uniform positive lower bound, it overcomes the additive tail $C/\log p_k$ in (3) for all sufficiently large p_k.

4 A uniform lower bound on $S(n)$ for candidate families

For the structured candidates (prime support $\{2, 3, \ldots, p_k\}$ and nonincreasing exponents) one has the uniform inequality

$$S(n) \;\geq\; \sum_{p \leq p_k} \frac{1}{p^2} \;\geq\; \frac{1}{4}. \tag{4}$$

Thus the factor $e^{-S(n)}$ compensates the tail $C/\log p_k$ beyond an effective range, leading to

$$\frac{\sigma(n)}{n} \;\leq\; e^{\gamma}\, \log p_k.$$

Using the swap argument (Appendix RH) to get $p_k < \log n$, we conclude

$$\frac{\sigma(n)}{n} < e^{\gamma}\, \mathrm{loglog}\, n,$$

which is Robin's inequality along these candidates. Stronger Robopol-type statements remain numerical here.

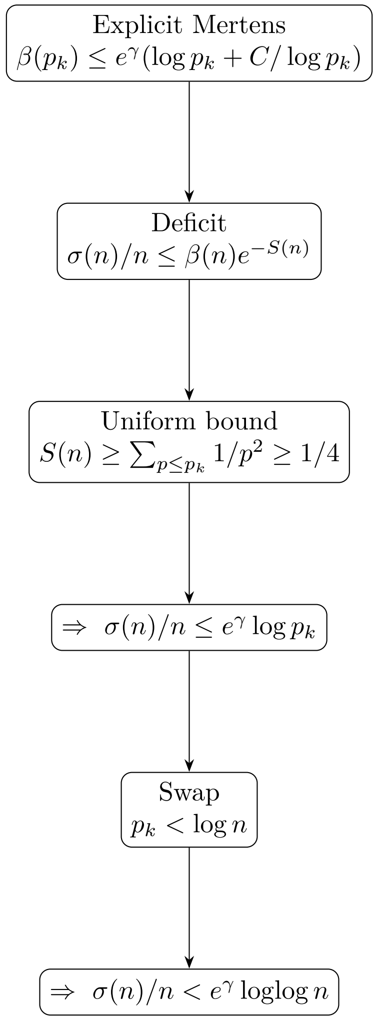

Figure 1: Logical pipeline in Appendix No. 2 combining explicit Mertens bounds with the multiplicative deficit.

Checklist for replication

1. Compute $\beta(n)$ up to p_k and $S(n)$ from the factorization of n.
2. Verify (2) numerically and check $\Xi(n) \le 1$.
3. Evaluate the right-hand side of (3) and confirm the margin provided by $e^{-S(n)}$.
4. Use $p_k < \log n$ (Appendix RH) to finalize $\sigma(n)/n < e^\gamma \operatorname{loglog} n$.

References

References

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